Factorization and entanglement in general XYZ spin arrays in non-uniform transverse fields

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We determine the conditions for the existence of a pair of degenerate parity breaking separable eigenstates in general arrays of arbitrary spins connected through XYZ couplings of arbitrary range and placed in a transverse field, not necessarily uniform. Sufficient conditions under which they are ground states are also provided. It is then shown that in finite chains, the associated definite parity states, which represent the actual ground state in the immediate vicinity of separability, can exhibit entanglement between any two spins regardless of the coupling range or separation, with the reduced state of any two subsystems equivalent to that of pair of qubits in an entangled mixed state. The corresponding concurrences and negativities are exactly determined. The same properties persist in the mixture of both definite parity states. These effects become specially relevant in systems close to the XXZ limit. The possibility of field induced alternating separable solutions with controllable entanglement side limits is also discussed. Illustrative numerical results for the negativity between the first and the $j^{\rm th}$ spin in an open spin s chain for different values of s and j are as well provided.

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Quantum entanglement constitutes one of the most fundamental, complex and counter-intuitive aspects of quantum mechanics. It is an essential resource in quantum information theory [1], playing a key role in quantum teleportation [2] and computation [1, 3, 4]. It also provides a deeper understanding of quantum correlations in many-body systems [5]. In particular, a great effort has been devoted in recent years to analyze entanglement and its connection with critical phenomena in spin chains [5, 6, 7, 8]. Studies of *finite* chains, of most interest for quantum information applications, are presently also motivated by the possibility of their controllable simulation through quantum devices [9, 10].

A remarkable feature of interacting spin chains is the possibility of exhibiting *exactly separable* ground states (GS) for special values of the external magnetic field, first discovered in [11, 12] in a $1D$ XYZ chain with first neighbor coupling. It was recently investigated in more general arrays under uniform fields [13, 14, 15, 16, 17, 18, 19], with a completely general method for determining separability introduced in [18]. Another remarkable related aspect is the fact that in the immediate vicinity of these separability points (SP) the entanglement between two spins can reach *infinite range* [15, 17]. In [17] we have shown that the SP in finite cyclic spin 1/2 arrays in a transverse field corresponds actually to a GS transition between opposite parity states (the last level crossing for increasing field), with the entanglement between *any* two spins reaching there finite side limits irrespective of the coupling range. In a small chain, this SP plays then the role of a "quantum critical point". In contrast, the entanglement range remains typically finite and low at the conventional phase transition [6].

The aim of this work is to generalize previous results to XYZ arrays of *arbitrary* spins and geometry in a *general* transverse field, not necessarily uniform. Moreover, we will also determine the exact side limits of the entanglement between *any* two subsystems (including those for the block entropy and those for any two spins or group of spins) at the SP analytically, for *any* spin value. A non-uniform field will be shown to allow exact separability with infinite entanglement range in its vicinity in quite diverse systems (such as open or non-uniform chains), including the possibility of *field induced alternating separable solutions* along separability curves, with *controllable* entanglement side limits. Illustrative results for the negativity between the first and $j^{\rm th}$ spin in an open spin s chain as a function of field and separation are as well presented, for different spin values.

We consider n spins $\boldsymbol{s}_i$ (which can be regarded as qudits of dimension $d_i = 2s_i + 1 \geq 2$) not necessarily equal, interacting through XYZ couplings of arbitrary range in the presence of a transverse external field b^i, not necessarily uniform. The Hamiltonian reads

$$H = \sum_i b^i s_i^z - \tfrac{1}{2}\sum_{i,j}(v_x^{ij} s_i^x s_j^x + v_y^{ij} s_i^y s_j^y + v_z^{ij} s_i^z s_j^z)\,, \quad (1)$$

and commutes with the global S_z parity or phase-flip $P_z = \exp[i\pi\sum_{i=1}^n (s_i^z + s_i)]$ for any values of b^i, v_μ^{ij} or s_i. Self-energy terms ($i = j$), non-trivial for $s_i \geq 1$, are for instance present in recent coupled cavity based simulations of arbitrary spin XXZ models [10] and will be allowed if $s_i \geq 1$.

We now seek the conditions for which such system will possess a *separable parity breaking eigenstate* of the form

$$|\Theta\rangle = \otimes_{i=1}^n \exp[i\theta_i s_i^y]|0_i\rangle \quad (2)$$

$$= \otimes_{i=1}^n\Big[\sum_{k=0}^{2s_i}\sqrt{\tbinom{2s_i}{k}}\cos^{2s_i-k}\tfrac{\theta_i}{2}\sin^k\tfrac{\theta_i}{2}|k_i\rangle\Big]\,, \quad (3)$$

where $s_i^z|k_i\rangle = (k - s_i)|k_i\rangle$ and $e^{i\theta_i s_i^y}|0_i\rangle$ is a rotated minimum spin state (coherent state [20]). The choice of y as rotation axis does not pose a loss of generality as

any state $e^{i\boldsymbol{\phi}_i\cdot\mathbf{s}_i}|0_i\rangle$ corresponds to a suitable complex θ_i in (2) [21]. Replacing s_i^μ in (1) by $e^{-i\theta_i s_i^y}s_i^\mu e^{i\theta_i s_i^y}$, i.e., $s_i^{z,x}\to s_i^{z,x}\cos\theta_i\pm s_i^{x,z}\sin\theta_i$, $s_i^y\to s_i^y$, the equation $H|\Theta\rangle=E_\Theta|\Theta\rangle$, i.e., $H_\Theta|0\rangle=E_\Theta|0\rangle$ with $|0\rangle=\otimes_{i=1}^n|0_i\rangle$ and $H_\Theta=e^{-i\sum_i\theta_i s_i^y}He^{i\sum_i\theta_i s_i^y}$, leads to the equations

$$v_y^{ij} = v_x^{ij}\cos\theta_i\cos\theta_j+v_z^{ij}\sin\theta_i\sin\theta_j\,, \tag{4}$$

$$b^i\sin\theta_i = \sum_j(s_j-\tfrac{1}{2}\delta_{ij})(v_x^{ij}\cos\theta_i\sin\theta_j-v_z^{ij}\sin\theta_i\cos\theta_j) \tag{5}$$

which determine, for instance, the values of v_y^{ij} and b^i in terms of v_x^{ij}, v_z^{ij}, s_i and θ_i. The energy is then given by

$$E_\Theta = -\sum_i s_i[b^i\cos\theta_i+\tfrac{1}{2}\sum_j(s_j-\tfrac{1}{2}\delta_{ij})(v_x^{ij}\sin\theta_i\sin\theta_j +v_z^{ij}\cos\theta_i\cos\theta_j)+\tfrac{1}{4}(v_x^{ii}+v_y^{ii}+v_z^{ii})]\,. \tag{6}$$

For a $1D$ spin s cyclic chain with first neighbor couplings ($v_\mu^{ij}=v_\mu\delta_{i,j\pm1}$) in a uniform field ($b^i=b$) we recover the original GS separability conditions of ref. [12] for both the ferromagnetic ($v_\mu\geq0$, $\theta_i=\theta$) and antiferromagnetic ($v_\mu\leq0$, $\theta_i=(-1)^i\theta$) cases. Eqs. (4)–(6) are however completely general and actually hold also for *complex* values of θ_i, v_μ^{ij} and b^i: If satisfied $\forall i,j$, H will have a separable eigenstate (2) with eigenvalue (6). If $\sin\theta_i\neq0$ for some i, this eigenvalue is *degenerate*: $|\Theta\rangle$ will break parity symmetry and therefore, the partner state

$$|-\Theta\rangle=P_z|\Theta\rangle=\otimes_{i=1}^n\exp[-i\theta_i s_i^y]|0_i\rangle\,, \tag{7}$$

will be an exact eigenstate of H as well, with the same energy (6). The points in parameter space where the states $|\pm\Theta\rangle$ become exact eigenstates correspond necessarily to the crossing of at least two opposite parity levels.

For real θ_i, Eq. (5) is just the stationary condition for the energy (6) at fixed b^i, v_μ^{ij}. The state (2) can thus be regarded as a mean field trial state, with Eq. (5) the associated self-consistent equation. Eq. (4), which is spin independent (at fixed v_μ^{ij}), ensures that it becomes an exact eigenstate by canceling the residual one and two-site matrix elements connecting $|\Theta\rangle$ with the remaining states. Moreover, if $\theta_i\in(0,\pi)\forall i$ and

$$|v_y^{ij}|\leq v_x^{ij}\quad\forall i,j\,, \tag{8}$$

we can ensure that $|\pm\Theta\rangle$ will be *ground states* of H: In the standard basis formed by the states $\{\otimes_{i=1}^n|k_i\rangle\}$, the terms in H depending on $\{s_i^z\}$ are diagonal whereas the rest lead to real non-positive off-diagonal matrix elements, as $\sum_{\mu=x,y}v_\mu^{ij}s_i^\mu s_j^\mu=\sum_{\nu=\pm}v_\nu^{ij}(s_i^+s_j^{-\nu}+s_i^-s_j^\nu)$, where $s_j^\pm=s_j^x\pm is_j^y$ and $v_\pm^{ij}=\frac{1}{4}(v_x^{ij}\pm v_y^{ij})\geq0$ by Eq. (8). Hence, $\langle H\rangle$ can be minimized by a state with all coefficients real and of the same sign in this basis (different signs will not decrease $\langle H\rangle$), which then, cannot be orthogonal to $|\Theta\rangle$ (Eq. (3)). With suitable phases for θ_i, $|\pm\Theta\rangle$ can also be GS in other cases: A π rotation around the z axis at site i leads to $\theta_i\to-\theta_i$ and $v_{x,y}^{ij}\to-v_{x,y}^{ij}$ for $i\neq j$.

Definite parity eigenstates of H in the subspace generated by the states $|\pm\Theta\rangle$ can be constructed as

$$|\Theta^\pm\rangle = \frac{|\Theta\rangle\pm|-\Theta\rangle}{\sqrt{2(1\pm O_\Theta)}}, \tag{9}$$

$$O_\Theta \equiv \langle-\Theta|\Theta\rangle=\textstyle\prod_{i=1}^n\cos^{2s_i}\theta_i\,, \tag{10}$$

which satisfy $P_z|\Theta^\pm\rangle=\pm|\Theta^\pm\rangle$, $\langle\Theta^\nu|\Theta^{\nu'}\rangle=\delta^{\nu\nu'}$. Here we have set θ_i real $\forall i$, since by local rotations around the z axis we can always choose y_i in the direction of $\boldsymbol{\phi}_i$ (and hence θ_i real) in the final state $|\Theta\rangle$. Moreover, we may also set $|\theta_i|\leq\pi/2$ (and hence $O_\Theta\geq0$) since a local rotation of π around the x axis leads to $\theta_i\to\pi-\theta_i$. The overlap (10) will play an important role in the following.

When the degeneracy at the SP is indeed 2, the states (9) (rather than (2)) are *the actual side limits at the SP* of the corresponding non-degenerate (and hence definite parity) exact eigenstates of H. For small variations δb^i, the degeneracy will be broken if $O_\Theta\neq0$, with an energy gap given by $\Delta E\approx\sum_i\delta b^i\Delta M_i$, where

$$\Delta M_i\equiv\langle\Theta^-|s_i^z|\Theta^-\rangle-\langle\Theta^+|s_i^z|\Theta^+\rangle=\frac{2s_i\sin^2\theta_i\,O_\Theta}{\cos\theta_i(1-O_\Theta^2)}\,.$$

(In contrast, $\langle\pm\Theta|s_i^z|\pm\Theta\rangle=-s_i\cos\theta_i$). When $|\Theta^\pm\rangle$ are GS, a GS parity transition $|\Theta^-\rangle\to|\Theta^+\rangle$, characterized by a *magnetization step* $\Delta M=\sum_i\Delta M_i$, will then take place at the SP if all or some of the fields are increased across the factorizing values (5). If ΔE or ΔM can be resolved or measured, the realization of the states (9) is then ensured. Their magnitude is governed by the overlap (10), appreciable in small systems (if $\theta_i\neq\pi/2$) as well as in finite systems with small angles $\theta_i^2\approx\delta_i/n$, such that $O_\Theta\approx e^{-\sum_i s_i\delta_i/n}$. This implies (Eq. (4)) systems close to the XXZ limit ($v_y^{ij}=v_x^{ij}$). In this limit ($\theta_i\to0$), $\Delta M\to1$, with $|\Theta^+\rangle\to|0\rangle$ and $|\Theta^-\rangle\propto\sum_i\sqrt{s_i}\theta_i|1_i\rangle$ (weighted W-type state), where $|1_i\rangle\equiv\otimes_{j=1}^n|(\delta_{ji})_j\rangle$.

Entanglement of definite parity states. In contrast with $|\pm\Theta\rangle$, the states (9) are entangled. If $\sin\theta_i\neq0\,\forall\,i$ the Schmidt number for *any* global bipartition $(A,\bar{A})$ is 2 and the Schmidt decomposition is

$$|\Theta^\pm\rangle = \sqrt{p^\pm_{A+}}|\Theta_A^+\rangle|\Theta_{\bar{A}}^\pm\rangle+\sqrt{p^\pm_{A-}}|\Theta_A^-\rangle|\Theta_{\bar{A}}^\mp\rangle\,, \tag{11}$$

$$p^\pm_{A^\nu} = \tfrac{(1+\nu O_A)(1\pm\nu O_{\bar{A}})}{2(1\pm O_\Theta)}\,,\quad O_A=\langle-\Theta_A|\Theta_A\rangle\,, \tag{12}$$

where $|\Theta_A^\pm\rangle$, $|\Theta_{\bar{A}}^\pm\rangle$ denote the analogous normalized definite parity states for the subsystems A, $\bar{A}$, with $\nu=\pm$, $O_AO_{\bar{A}}=O_\Theta$ and $p^\pm_{A+}+p^\pm_{A-}=1$. Hence, $|\Theta^\pm\rangle$ can be effectively considered as *two qubit states* with respect to *any* bipartition $(A,\bar{A})$, with $|\Theta_A^\pm\rangle$, $|\Theta_{\bar{A}}^\pm\rangle$ representing the orthogonal states of each qubit. Accordingly, the reduced density matrix $\rho_A^\pm$ of subsystem A in the state $|\Theta^\pm\rangle$ is

$$\rho_A^\pm=p^\pm_{A+}|\Theta_A^+\rangle\langle\Theta_A^+|+p^\pm_{A-}|\Theta_A^-\rangle\langle\Theta_A^-|\,. \tag{13}$$

The entanglement between A and its complement $\bar{A}$ can be measured through the global concurrence (square

root of the tangle [22]) $C_{A\bar{A}} = \sqrt{2(1-\mathrm{tr}\,\rho_A^2)}$, which for a rank 2 density is just an increasing function of the entanglement entropy $E_{A\bar{A}} = -\mathrm{tr}\,\rho_A \log_2 \rho_A$, with $C_{A\bar{A}} = E_{A\bar{A}} = 0$ (1) for a separable (Bell) state. In the states (11) we then obtain

$$C^{\pm}_{A\bar{A}} = \frac{\sqrt{(1-O_A^2)(1-O_{\bar{A}}^2)}}{1 \pm O_\Theta}\,. \tag{14}$$

These values represent the side limits of $C_{A\bar{A}}$ at the SP. For $O_\Theta > 0$, $C^-_{A\bar{A}} > C^+_{A\bar{A}}$, with $C^-_{A\bar{A}} = 1$ if $O_A = O_{\bar{A}}$. Note that $|\Theta^\pm\rangle$ are simultaneous Bell states for $(A, \bar{A})$ only if $O_A = O_{\bar{A}} = 0$ (GHZ limit of $|\Theta^\pm\rangle$). Increasing overlaps will in general decrease the global entanglement.

At the SP, the entanglement entropy of a block of L spins in a $1D$ first neighbor spin 1/2 XY chain in a constant field was found in [23] to be $S_L = -\mathrm{tr}\rho_L \ln \rho_L = \ln 2$ (i.e., $C_{L\bar{L}} = E_{L\bar{L}} = 1$) in the thermodynamic limit, in agreement with Eq. (14) for vanishing overlaps. Eq. (14) extends this result to general *finite* chains, leading to a slightly smaller value: For small O_A, $O_{\bar{A}}$, $C^{\pm}_{A\bar{A}} \approx 1 - \frac{1}{2}(O_A \pm O_{\bar{A}})^2$ and $S^{\pm}_L \approx \ln 2 - \frac{1}{2}(O_L \pm O_{\bar{L}})^2$ (with $O_L = (\frac{v_y}{v_x})^{\frac{L}{2}}$ in the $s = 1/2$ XY chain).

Pairwise and subsystem entanglement. On the other hand, the entanglement of a subsystem is enabled by non-zero overlaps. A remarkable feature of the states (9) is that *any* two spins or disjoint subsystems B, C *will also be entangled* if the complementary overlap $O_{\overline{B+C}}$ is non-zero and $O_B^2 < 1$, $O_C^2 < 1$. Moreover, this entanglement can be characterized by the concurrence

$$C^{\pm}_{BC} = \frac{\sqrt{(1-O_B^2)(1-O_C^2)}O_{\overline{B+C}}}{1 \pm O_\Theta}\,, \tag{15}$$

or equivalently, the negativity [24, 25],

$$N^{\pm}_{BC} = \tfrac{1}{2}\left[\sqrt{(p^{\pm}_{A^{\mp}})^2 + (C^{\pm}_{BC})^2/O_{\overline{B+C}}} - p^{\pm}_{A^{\mp}}\right], \tag{16}$$

where $A = B + C$. While the concurrence of an arbitrary mixed state ρ_A (which can be defined through the convex roof extension of the pure state definition [26]) is not directly computable in general (the exception being the case of two qubits [27]), the negativity $N_{BC} = \frac{1}{2}[\mathrm{Tr}|\rho_A^{t_B}| - 1]$, where $\rho_A^{t_B}$ denotes partial transpose [28], can always be calculated [29], being just the absolute value of the sum of the negative eigenvalues of $\rho_A^{t_B}$. Eq. (16) represents then the side-limits of N_{BC} at the SP.

Proof: For $A = B + C$, we first note that if $O_{\bar{A}} = 0$, Eq. (13) becomes $\rho_A^{\pm} = \frac{1}{2}(|\Theta_A\rangle\langle\Theta_A| + |-\Theta_A\rangle\langle-\Theta_A|)$, i.e., $\rho_A^{\pm}$ *coincident and separable* (convex combination of product densities [30]). Entanglement between B and C can then only arise if $O_{\overline{B+C}} \neq 0$. Next, using similar Schmidt decompositions (11) of the states $|\Theta_A^{\pm}\rangle$, Eq. (13) can also be considered as an *effective two-qubit mixed state* with respect to *any* bipartition (B, C) of A: Its support will lie in the subspace spanned by the four states $\{|\Theta_B^{\nu}\rangle|\Theta_C^{\nu'}\rangle\,, \nu, \nu' = \pm\}$, such that

$$\rho_A^{\pm} = \begin{pmatrix} p^{\pm}_{A^+}q^{+}_{BC^+} & 0 & 0 & p^{\pm}_{A^+}\alpha^{+}_{BC} \\ 0 & p^{\pm}_{A^-}q^{-}_{BC^+} & p^{\pm}_{A^-}\alpha^{-}_{BC} & 0 \\ 0 & p^{\pm}_{A^-}\alpha^{-}_{BC} & p^{\pm}_{A^-}q^{-}_{BC^-} & 0 \\ p^{\pm}_{A^+}\alpha^{+}_{BC} & 0 & 0 & p^{\pm}_{A^+}q^{+}_{BC^-} \end{pmatrix}$$

where $q^{\pm}_{BC^\nu} = \frac{(1+\nu O_B)(1\pm\nu O_C)}{2(1\pm O_B O_C)}$, $\alpha^{\pm}_{BC} = \sqrt{q^{\pm}_{BC^+}q^{\pm}_{BC^-}}$ and $q^{\pm}_{BC^+} + q^{\pm}_{BC^-} = 1$. $\rho_A^{\pm}$ will be entangled if its partial transpose has a negative eigenvalue [28], a condition here equivalent to a positive mixed state concurrence [27] $C^{\pm}_{BC} = \mathrm{Max}[C^{\pm}_+, C^{\pm}_-, 0]$, where $C^{\pm}_\nu = 2[p^{\pm}_{A^\nu}\alpha^{\nu}_{BC} - p^{\pm}_{A^{-\nu}}\alpha^{-\nu}_{BC}]$ represent parallel ($\nu = +$) or antiparallel ($\nu = -$) concurrences, i.e. driven by $|\Theta_A^+\rangle$ or $|\Theta_A^-\rangle$ in Eq. (13). This leads to Eq. (15), with C^+_{BC} (C^-_{BC}) *parallel* (*antiparallel*). The ensuing negativity, given here by minus the negative eigenvalue of the partial transpose $(\rho_A^{\pm})^{t_B}$, is then given by Eq. (16).

For $B = A$, $C = \bar{A}$ ($O_{\overline{B+C}} \to 1$), Eq. (15) reduces to (14), with $N^{\pm}_{A\bar{A}} = \frac{1}{2}C^{\pm}_{A\bar{A}}$. For a pair of spins $i \neq j$, $O_B = \cos^{2s_i}\theta_i$, $O_C = \cos^{2s_j}\theta_j$ and the result of [17] is recovered from (15) if $s_i = \frac{1}{2}$ and $\theta_i = \theta\ \forall i$. We finally note that if $O_B = O_C$, $N^+_{BC} = C^+_{BC}/2$, as in the case of a global partition. In general, however, there is no proportionality between $N^{\pm}_{BC}$ and $C^{\pm}_{BC}$.

The concurrences (15) fulfill the monogamy inequalities [31] $C^2_{B,C+D} \geq C^2_{BC} + C^2_{BD}$ for any three disjoint subsystems B, C, D. We actually obtain here

$$C^2_{BC} + C^2_{BD} = C^2_{B,C+D}\left[1 - \tfrac{(1-O_C^2)(1-O_D^2)}{1-O_C^2 O_D^2}\right]. \tag{17}$$

Let us also remark that subsystem entanglement persists, though attenuated, in the uniform mixture

$$\rho^0 = \tfrac{1}{2}(|\Theta^+\rangle\langle\Theta^+| + |\Theta^-\rangle\langle\Theta^-|)\,, \tag{18}$$

which differs from $\frac{1}{2}(|\Theta\rangle\langle\Theta| + |-\Theta\rangle\langle-\Theta|)$ if $O_\Theta \neq 0$ and represents the $T \to 0^+$ limit of the thermal state $\rho \propto e^{-\beta H}$ at the SP when $|\pm\Theta\rangle$ are GS (and the GS degeneracy there is 2). Replacing $p^{\pm}_{A^\nu}$ by $\frac{1}{2}(p^+_{A^\nu} + p^-_{A^\nu})$ in (13), we find now *antiparallel global and subsystem concurrences*, given for any disjoint subsystems B, C by

$$C^0_{BC} = \tfrac{1}{2}(C^-_{BC} - C^+_{BC}) = C^-_{BC}O_\Theta/(1 + O_\Theta)\,, \tag{19}$$

i.e., half the parity splitting of C_{BC}. Eq. (19) remains valid for a global bipartition ($B = A$, $C = \bar{A}$). The ensuing negativity can be similarly calculated.

The order of magnitude of subsystem concurrences is governed by the complementary overlap $O_{\overline{B+C}}$. For small subsystems (like a pair of spins) in a large system, $C^{\pm}_{BC}$ will be appreciable just for sufficiently small angles in the complementary system, i.e., $\theta_i^2 \approx \delta_i/n$, such that $O_{\overline{B+C}} \approx e^{-\sum_{i\in\bar{A}} s_i\delta_i/n}$ remains finite. This leads again to systems with small XY anisotropy.

Uniform Solution. Let us now examine the possibility of a common angle $\theta_i = \theta \; \forall \; i$. Eq. (4) leads then to

$$v_y^{ij} - v_z^{ij} = (v_x^{ij} - v_z^{ij}) \cos^2\theta \,, \tag{20}$$

implying a fixed ratio $\chi \equiv (v_y^{ij} - v_z^{ij})/(v_x^{ij} - v_z^{ij}) = \cos^2\theta$ for *all* pairs with $v_x^{ij} \neq v_z^{ij}$, and an isotropic coupling $v_y^{ij} = v_x^{ij}$ if $v_x^{ij} = v_z^{ij}$. A subset of isotropic couplings will not spoil this eigenstate [32]. Eq. (5) implies then b^i arbitrary if $\theta = 0$ or π (XXZ case $v_y^{ij} = v_x^{ij}$) or otherwise

$$b^i = \cos\theta \sum_j (v_x^{ij} - v_z^{ij})(s_j - \tfrac{1}{2}\delta_{ij}). \tag{21}$$

The energy (6) becomes

$$E_\Theta = -\tfrac{1}{2}\sum_{i,j} s_i[s_j(v_x^{ij} + v_y^{ij} - v_z^{ij}) + \delta_{ij} v_z^{ii}]\,. \tag{22}$$

A general field allows then a uniform separable eigenstate (a global coherent state) in cyclic as well as open chains with arbitrary spins s_i in any dimension if (20) holds $\forall i, j$. For instance, in an open $1D$ spin s chain with first neighbor couplings $v_\mu^{ij} = v_\mu \delta_{i,j\pm 1}$, Eq. (21) yields $b^i = b_s = 2s\sqrt{(v_y - v_z)(v_x - v_z)}$ at inner sites but $b^1 = b^n = \frac{1}{2} b_s$ *at the borders.*

Eqs. (20)–(22) are actually valid for general complex θ, but real fields imply $\cos\theta$ real ($\chi \geq 0$). The case $\cos^2\theta > 1$ (imaginary θ) corresponds to a rotation around the x axis but can be recast as a rotation around the y axis by a global rotation around the z axis. Hence, we may set $\cos^2\theta \in [0, 1]$. $|\pm\Theta\rangle$ will then be GS when Eq. (8) holds.

The concurrence (15) becomes, setting $\cos^2\theta = \chi$,

$$C_{BC}^{\pm} \;=\; \frac{\sqrt{(1 - \chi^{2S_B})(1 - \chi^{2S_C})}\,\chi^{S - (S_B + S_C)}}{1 \pm \chi^S} \tag{23}$$

where $S_B = \sum_{i \in B} s_i$ is the subsystem total spin and $S = \sum_i s_i$ the total spin. It is independent of separation and coupling range, depending solely on χ^S and the ratios S_B/S, S_C/S. If $\chi = 1 - \delta/2S$, with $\delta > 0$ and finite, $\chi^S \approx e^{-\delta/2}$ remains finite for large S. Eq. (23) leads then to $O(1/\sqrt{S})$ and $O(1/S)$ global and subsystems concurrences for small S_A, S_B and S_C:

$$C_{A\bar{A}}^{\pm} \;\approx\; \sqrt{\frac{S_A \delta}{S}}\frac{\sqrt{1 - e^{-\delta}}}{1 \pm e^{-\delta/2}}\,, \tag{24}$$

$$C_{BC}^{\pm} \;\approx\; \frac{\delta}{S}\frac{\sqrt{S_B S_C}\, e^{-\delta/2}}{1 \pm e^{-\delta/2}}\,. \tag{25}$$

On the other hand, for $S_A = \frac{1}{2}S$, $C_{A\bar{A}}^- = 1$ whereas $C_{A\bar{A}}^+ = \tanh\frac{1}{4}\delta$. Thus, while for large δ both $C_{A\bar{A}}^{\pm}$ rapidly approach 1 as S_A increases, for small δ (XXZ limit) this occurs just for $C_{A\bar{A}}^-$ and S_A close to $S/2$ (here $|\Theta^+\rangle \to |0\rangle$ but $|\Theta^-\rangle$ approaches the W-type state $\propto \sum_i \sqrt{s_i}|1_i\rangle$).

Alternating solution and controllable entanglement at the SP. Among other possibilities allowed by Eqs. (4)–(5), let us examine that of a *field induced two-angle solution* in a $1D$ chain (cyclic or open) of spin s with first neighbor XY couplings ($v_\mu^{ij} = \delta_{i,j\pm 1} v_\mu$, with $v_z = 0$). We assume $\chi = v_y/v_x \in [0, 1]$. A separable eigenstate with $\theta_{2i} = \theta_e$, $\theta_{2i-1} = \theta_o$ is feasible if there is an alternating field $b^{2i} = b_e$, $b^{2i-1} = b_o$ in inner sites satisfying (Eqs. (4)–(5))

$$b_e b_o = (2s)^2 v_x v_y\,. \tag{26}$$

This leads to a transverse *separability curve.* The ensuing angles satisfy $\cos\theta_o \cos\theta_e = v_y/v_x$ and are given by

$$\cos^2\theta_\sigma = \frac{b_\sigma^2 + (2sv_y)^2}{b_\sigma^2 + (2sv_x)^2}\,, \;\; \sigma = o, e\,, \tag{27}$$

being *field dependent.* For $b_e = b_o$ we recover the previous uniform solution ($b_s = 2s\sqrt{v_x v_y}$). In an open chain we should just add, according to Eq. (5), the border corrections $b^1 = \frac{1}{2}b_o$, $b^n = \frac{1}{2}b_{\sigma_n}$. The states $|\pm\Theta\rangle$ will then be GS setting $\theta_{o,e} > 0$ when $v_x > 0$ and $\theta_o > 0$, $\theta_e < 0$ in the antiferromagnetic case $v_x < 0$ (for even n if chain is cyclic, to avoid frustration).

The definite parity states $|\Theta^\pm\rangle$ will again lead to infinite entanglement range, but with three different field dependent (and hence *controllable*) pairwise concurrences between any two spins (Eq. (15) for $B = i$, $C = j$): even-even, odd-odd and even-odd, satisfying $C_{oe}^{\pm} = \sqrt{C_{oo}^{\pm} C_{ee}^{\pm}}$, with $C_{oo}^{\pm} > C_{oe}^{\pm} > C_{ee}^{\pm}$ if $|b_o| < |b_e|$. Hence, $C_{oo}^{\pm}$ *can be made larger than* $C_{oe}^{\pm}$ despite the absence of odd-odd direct coupling. For sufficiently large b_e, $\cos\theta_e \approx 1$ but $\cos\theta_o \approx \chi$: just odd-odd pairs will be appreciably entangled in this limit at the SP.

Application. As illustration, we first depict in fig. 1 full exact results for the GS negativities N_{1j} between spins 1 and j in a small open chain of uniform spin s with first neighbor XY couplings in a uniform transverse field $b^i = b$ for $i = 2, \ldots, n-1$, with the border corrections $b^1 = b^n = \frac{1}{2}b$. For $\chi = v_y/v_x \in (0, 1)$ this chain will then exhibit an exact factorizing field $b_s = 2sv_x\sqrt{\chi}$ where separable parity breaking states with uniform angle $\cos\theta = \sqrt{\chi}$ will become exact GS if $v_x > 0$ (if $v_x < 0$, $\theta_i = (-1)^i\theta$ instead in the GS). We have set $\chi = 1 - \delta/(2ns)$, such that the side limits of the negativity at b_s are roughly independent of s and n. It is first seen that the ensuing behavior of the N_{1j} in terms of the scaled field b/b_s is quite similar for the three spin values considered ($s = 1/2$, 1 and 3/2, the latter involving a diagonalization in a basis of 65536 states for $n = 8$). The GS exhibits ns parity transitions as the field is increased from 0^+ to b_s, with the last transition at b_s. As the latter is approached, it is verified that the pairwise entanglement range increases, with *all* negativities approaching the common side limits (16), distinct at each side, given here by $N_{ij}^+ = \frac{1}{2}C_{ij}^+ \approx \frac{\delta e^{-\delta/2}}{2n(1 + e^{-\delta/2})}$ (Eq. (25)) and $N_{ij}^- \approx \frac{(C_{ij}^-)^2 e^{\delta/2}}{4p_{A^-}^-} \approx \frac{\delta^2 e^{-\delta/2}}{4n^2(1 - e^{-\delta/2})^2}$. An interval of full range pairwise entanglement around b_s is then originated, which involves on the left side essentially the last state

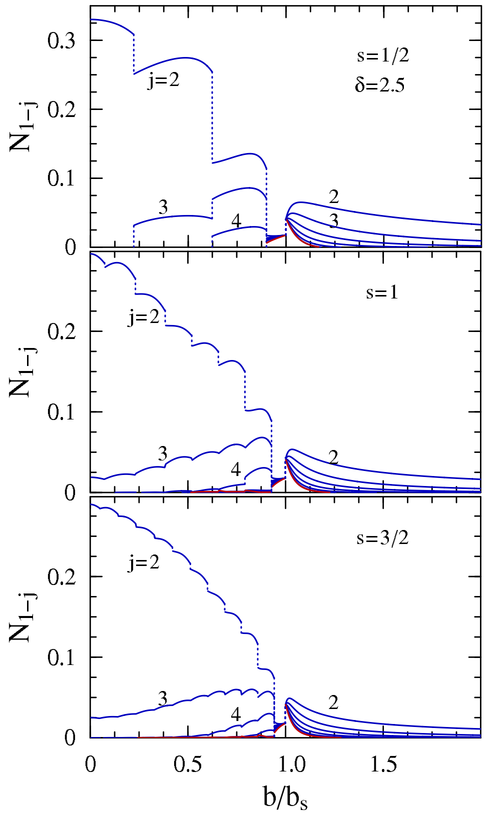

FIG. 1: (Color online) Negativities between the first and the $j^{\rm th}$ spin in an open spin s chain with first neighbor XY coupling, as a function of the transverse field b, with border corrections (see text), and three different values of s. We have set an anisotropy $v_y/v_x = 1 - \delta/(2sn)$, with $\delta = 2.5$ and $n = 8$ spins. The factorizing field corresponds to the last parity transition, and is singled out as the field where all negativities merge, approaching common non-zero distinct side limits. The lowest line (in red) depicts the end-to-end negativity (N_{1-n}).

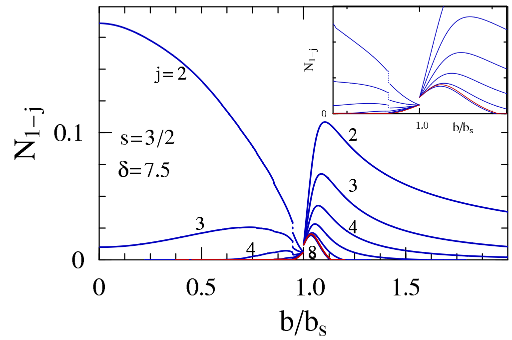

FIG. 2: (Color online) Same details as fig. 1 for $\delta = 7.5$ and $s = 3/2$. The inset depicts the behavior in the vicinity of the separability field.

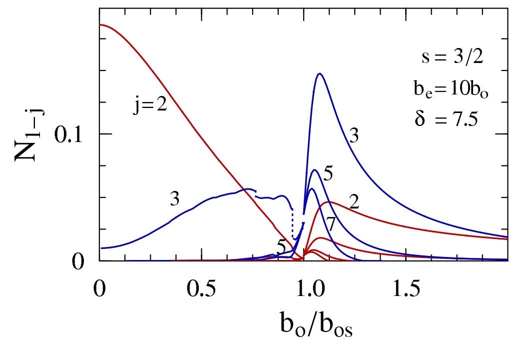

FIG. 3: (Color online) Same details as fig. 1 for $\delta = 7.5$, $s = 3/2$ and an alternating field with fixed even/odd ratio $b_e/b_o = 10$. Red (blue) lines depict results for j even (odd).

before the last transition (roughly an W-state). b_s plays in this small chain the role of a quantum critical field.

Fig. 2 depicts results for a greater anisotropy $\delta = 7.5$. In this case just the last two parity transitions are visible in the negativity. The common side limits of $N_{ij}^{\pm}$ are smaller and all negativities exhibit a maximum to the right of the factorizing field. The behavior is then more similar to that of larger XY systems [15]. Nonetheless, there is still a clear interval of full entanglement range around b_s, with finite side-limits at b_s when observed in detail (inset).

The side limits at separability can actually be modified in this system by changing the even-odd field ratio $\eta = b_e/b_o$, according to Eq. (26). Results for a fixed ratio $\eta = 10$ (with pertinent border corrections) are shown in Fig. 3, in which case separability is exactly attained at an odd field $b_{os} = b_s/\sqrt{\eta}$. We have again plotted just the negativities between the first and the j spin, which now approach *two* common side limits at each side, one for j even ($N_{oe}^{\pm}$) and one for j odd ($N_{oo}^{\pm}$). While the former become quite small, the latter become clearly appreciable, the final effect for such large ratios being essentially that just odd sites become uniformly entangled in the vicinity of b_{os}. Even-even negativities $N_{ee}^{\pm}$ (not shown) are of course also very small at b_{os}. Notice finally that N_{13} can become much larger than N_{12} in the region around b_{os}, despite the absence of second neighbor couplings.

In summary, we have first determined the conditions for the existence of separable parity breaking (and locally coherent) eigenstates in general XYZ arrays of arbitrary spins in a general transverse field, showing in particular the possibility of exact separability in open as well as non-uniform chains through non-uniform transverse fields. We have also determined the entanglement properties of the associated definite parity states, through the evaluation of the concurrence and negativity for any pair of spins or subsystems, for any spin values. These states, which approach both GHZ and W-states in particular limits, exhibit full entanglement range when non-orthogonal, and can be seen as effective two qubit entangled states for any bipartition. Moreover, the same holds for their uniform mixture as well as for the reduced density of any subsystem. The finite entanglement limits at the SP become relevant in finite arrays close to the XXZ limit, where the separability field can be clearly identified with the last GS parity transition, as verified in the nu-

merical results presented, playing the role of a quantum critical field. The possibility of exact separability in an alternating field ($b_e = \eta b_o$) for arbitrary even-odd ratios η, leading to controllable entanglement side-limits, has also been disclosed. The present results provide a deeper understanding of the behavior of pairwise entanglement in finite XYZ spin arrays subject to transverse fields.

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